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The analysis considered in this paper is applicable to many linear systems, such as linear forms, matrices, vectors, etc. For convenience of language it is here stated in terms of linear spaces.

A theorem involving in its statement only linear spaces can be expressed in terms of two processes, the determination of the space of certain spaces and the determination of the space common to certain spaces. It may involve two relations, of spaces contained in a space of a certain order and of spaces intersecting in a space of a certain order. We develop, after Grassmann, an analysis which expresses symbolically these relations and the results of these processes.

If a, b, c are points and λ, μ, ν numbers,

$$\lambda a + \mu b + \nu c,$$

interpreted according to matrix theory, represents a point in the space of a, b, c and, if λ, μ, ν take all values, represents all points in that space. This expression vanishes only when the points lie in a lower space. Thus with undetermined multipliers we can express the relations desired. This method is usually very clumsy, however, and the peculiar excellence of Grassmann's system consists in replacing these sums with unknown coefficients by products without them.

For this purpose we consider what Gibbs called an indeterminate product. It has the following properties:

$$(1) \quad A + B = B + A,$$
$$(2) \quad (A + B) + C = A + (B + C),$$
$$(3) \quad A(B + C) = AB + AC,$$
$$(4) \quad (AB)C = A(BC),$$
$$(5) \quad \lambda A = A\lambda,$$
$$(6) \quad 0A = A0 = 0,$$

[1] The method discussed in this paper was suggested by Gibbs in his vice-presidential address before the American Association for the Advancement of Science (Scientific Papers, **2**, 109).

where A, B, C are multiple quantities or numbers and λ a number. An equality that is a consequence of these relations alone will be called an identity.

We assume a space containing all the points of a given discussion. Points in this space are expressible as linear functions (in the matrix sense) of others. In the statement of a linear problem these relations between points are the only ones that occur. Hence an expression will be provably (and therefore defined as) zero only when it is reduced by linear relations identically to zero.

Let $f(a_1, a_2 \ . \ . \ . \ a_n)$ be a rational integral function of the n points a_i and

$$f(a_1 a_2 \ . \ . \ . \ a_n) = 0$$

There must then (by definition) be a set of points $b_1 \ . \ . \ . \ b_m$, in terms of which the a's are linearly expressible, such that

$$f(a_1 \ . \ . \ . \ a_n) = \phi(b_1 \ . \ . \ . \ b_m) \equiv 0.$$

If any of the b's are linear functions of the others we may suppose them replaced so that $b_1 \ . \ . \ . \ b_m$ are linearly independent. If $n > m$, the a's (expressible in terms of a smaller number of points) must satisfy a linear relation. If $n = m$ and the a's are linearly independent, the equations expressing them in terms of b's can be solved and the points b_i expressed in terms of the a's. Then, since an identity transforms into an identity,

$$\phi(b_1 \ . \ . \ . \ b_m) = f(a_1 \ . \ . \ . \ a_n) \equiv 0.$$

If $n < m$ and the a's linearly independent, the equations expressing a's in terms of b's can be solved for n b's, and since that part of $\phi(b_1 \ . \ . \ . \ b_m)$ containing these must vanish identically, $f(a_1 \ . \ . \ . \ a_n) \equiv 0$ as before. Thus if $f(a_1 \ . \ . \ . \ a_n)$ really contains the a's, these points can not be linearly independent. If they are replaced by any linearly independent set of points b_i, the new expression, since it vanishes, must vanish identically. Therefore, if a rational integral function of n points vanishes, these points satisfy a linear relation, and the given function reduces identically to zero when the n points are expressed as linear functions of *any* set of linearly independent points.

To form expressions that do vanish when the points are linearly related we use ordered determinants, i.e., determinants expanded like ordinary determinants with all products ordered, first term being taken from first column, second term from second column, etc. Thus,

$$\begin{vmatrix} a_1 \, a_1 \\ a_2 \, a_2 \end{vmatrix} = a_1 \, a_2 - a_2 \, a_1.$$

Such a determinant has certain properties of ordinary determinants. Thus, if two rows are equal, the determinant is zero; its value is not changed when a multiple of one row is added to another row; it can be expanded in terms of minors taken from first m columns. As the above example shows rows and columns are not interchangeable.

The necessary and sufficient condition that $a_1 \ldots a_n$ satisfy a linear relation is

$$\begin{vmatrix} a_1 \, a_1 \ldots a_1 \\ a_2 \, a_2 \ldots a_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n \, a_n \ldots a_n \end{vmatrix} = 0.$$

For if any expression vanishes, we have shown that the a's are linearly related. Conversely, if they satisfy a linear relation, since one row is a linear function of the others, the determinant is zero.

We represent the above determinant by the symbols $[a_1 \ldots a_n]$ or $(a_1 \ldots a_n)$. This may be regarded as a product of the points $a_1 \ldots a_n$. Gibbs called it the combinatorial product because it has the property (characteristic of Grassmann's combinatorial) of changing sign when two of the a's are interchanged.

Similarly we define a combinatorial product

$$[f(a_1 \ldots a_n) \, \phi(b_1 \ldots b_m)],$$

when f and ϕ are any rational integral functions. The two expressions are multiplied distributively and each product replaced by the sum of all permutations which leave the order of the a's and the order of the b's unchanged, the sign being negative when the permutation is odd. Thus the combinatorial product of a_1a_2 and b_1b_2 is

$$a_1a_2b_1b_2 + a_1b_1b_2a_2 + b_1b_2a_1a_2$$
$$+ b_1a_1a_2b_2 - a_1b_1a_2b_2 - b_1a_1b_2a_2.$$

From the preceding definition it follows that

$$[(a_1 \ldots a_n) \, (a_{m+1} \ldots a_n)] = (a_1 \ldots a_m).$$

For the left hand member expresses that every permutation of $a_1 \ldots a_m$ is to be placed in every position among the letters of every

permutation of $a_{m+1} \ldots a_n$. That is equivalent to the right side which represents all permutations of $a_1 \ldots a_n$ and the rule of sign is the same for both. It should be noted that $(a_1 a_2 b_1 b_2)$ is the combinatorial product of $(a_1 a_2)$ with $(b_1 b_2)$ and not of $a_1 a_2$ with $b_1 b_2$.

Let $A = (a_1 \ldots a_n) \neq 0$. If x is a linear function of $a_1 \ldots a_{n1}$, i.e., if x lies in the space of $a_1 \ldots a_n$,

$$(Ax) = (a_1 \ldots a_n x) = 0.$$

Hence $(Ax) = 0$ is the equation of that space and we may represent the space by the symbol A.

If

$$A = (a_1 \ldots a_n), \quad B = (b_1 \ldots b_m),$$
$$(AB) = (a_1 \ldots a_n b_1 \ldots b_m).$$

If the points $a_1 \ldots a_n b_1 \ldots b_m$ are independent, i.e., if the spaces do not intersect, (AB) represents the space determined by A and B. If the spaces A and B intersect, $(AB) = 0$. Because of this property of vanishing under incidence (property characteristic of Grassmann's progressive product) Gibbs called this product progressive. The progressive product represents our first linear process and by its vanishing gives the first linear relation.

We can now express the condition that two spaces intersect. To distinguish different types or degrees of incidence we need other products. We define as the regressive product in a space of order p the result of multiplying two functions distributively and replacing each term of the result by the sum of terms gotten by permuting, as in the progressive product, the first p letters in each term. If $A = (a_1 \ldots a_n)$, $B = (b_1 \ldots b_m)$

$$(AB)_p = \begin{vmatrix} a_1 a_1 & \ldots & a_1 0 & \ldots & 0 \\ a_2 a_2 & \ldots & a_2 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \\ a_n a_n & \ldots & a_n 0 & \ldots & 0 \\ b_1 b_1 & \ldots & b_1 & \ldots & b_1 \\ \ldots & \ldots & \ldots & & \\ b_m b_m & \ldots & b_m & \ldots & b_m \end{vmatrix}$$

there being p elements not zero in each row of the A's.

If $b_i \ldots b_j$ is any combination of b's, $b_k \ldots b_l$ the remaining ones,

$$B = \Sigma \pm (b_i \ldots b_j)(b_k \ldots b_l),$$

the sign being minus when

$$b_i \ldots b_j b_k \ldots b_l$$

is an odd permutation of $b_1 \ldots b_m$.

Multiplication by A gives

$$(AB)_p = \Sigma \pm (Ab_i \ldots b_j)(b_k \ldots b_l).$$

This is the fundamental identity of Grassmann,[2] many special cases of which are used in geometry.

Let A and B be two spaces intersecting in the space

$$C = (c_1 \ldots c_k).$$

We can determine $n-k$ points a_i and $m-k$ points b_i such that

$$A = (a_1 \ldots a_{n-k} c_1 \ldots c_k), \quad B = (b_1 \ldots b_{m-k}\, c_1 \ldots c_k).$$

We expand the product (AB) by the formula given above. If $p > m + n - k$, there are two c's equal in each of the prefactors and the result is zero. If $p = m + n - k$, there is just one term that does not vanish, giving

$$\begin{aligned}(AB)_p &= (a_1 \ldots a_{n-k} c_1 \ldots c_k b_1 \ldots b_{m-k})(c_1 \ldots c_k) \\ &= DC,\end{aligned}$$

where D is the space containing both A and B and C the space common to them. If $p < m + n - k$, there are a number of terms in the expansion. The determinants which are prefactors are all different. The same is true of postfactors. Since the points in it are linearly independent, the expression can not factor or vanish. Hence if $A \neq 0$, $B \neq 0$, $[AB]_p = 0$ is the necessary and sufficient condition that A and B lie in a space of order less than p. If the containing space is of order p, $[AB]_p$ is the product of that space and the common space.

Progressive products are used when the number of factors is equal to or less than p, regressive when that number is greater than p. Hence it causes no confusion to use the same notation $[AB]$ for both.

Expressions occurring in the discussions of linear geometry are always homogeneous. Upon multiplying two such expressions regressively, each term of the result is either zero or of the form DC where D is the space in which we are working. Since it is a factor of all

[2] Ausdehnungslehre (1862), p. 83.

such expressions, the equations connecting them are not changed by assuming D scalar. This amounts to making all progressive products of p letters scalar.

The regressive product represents the space common to two spaces or vanishes when they are contained in a space of order less than p. This product represents symbolically the second fundamental linear process, and by its vanishing the second type of relation.